Barriers for Tensor Scaling

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joint work with Cole Franks, [FR21], CCC 2021, arXiv:2102.06652

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Algebraic Geometry and Complexity Theory

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Outline

1. Introduction: setting, computational problems, motivation
    - Example: Matrix Scaling, Array Scaling
2. Scaling Algorithms and their Complexity
3. Bounds on Complexity Parameters
4. Glimpse of main Proof Techniques
5. Outlook and Open Problems

General Setting

- G is a reductive algebraic group over $\mathbb{C}$
  - e.g., $\mathrm{GL}(n)$, $(\mathbb{C}^\times)^n$, $\mathrm{SL}(n)$ and $\mathrm{ST}(n)$; products of these
- $\pi\colon G \to \mathrm{GL}(V)$ some finite-dim'l rational representation of G

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Concrete Example for this talk:

- *commutative:* $T = \mathrm{ST}(n)^d = \mathrm{ST}(n) \times \cdots \times \mathrm{ST}(n)$ (d copies)
- *non-commutative:* $G = \mathrm{SL}(n)^d$
- $V = (\mathbb{C}^n)^{\otimes d} = \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n$ (think of $d = 2$ or $d = 3$)
- $\pi_{n,d}\colon G \to \mathrm{GL}(V)$ representation of G given by the action

$$(g_1, \ldots, g_d) \cdot v \mapsto (g_1 \otimes \cdots \otimes g_d)(v)$$

Computational Problems

- For fixed $v \in V$, consider $F_v \colon G \to \mathbb{R},\ g \mapsto \|g \cdot v\|^2$.
- The ***capacity*** of v is
  $\operatorname{cap}_G(v) := \inf_{g \in G} F_v(g) = \inf_{g \in G} \|g \cdot v\|^2$.

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- **Norm Minimization:** Given $v \in V$ and $\varepsilon > 0$.
  Compute $g \in G$ such that $\|g \cdot v\|^2 \leq \operatorname{cap}_G(v) + \varepsilon$.
  - G commutative: *convex* optimization (geometric programming)
  - G non-commutative: *geodesic* convex optimization
  - **Desirable:** high precision efficiently, i.e., $\operatorname{poly}\big(n, \log(1/\varepsilon)\big)$

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  - **Desirable:** high precision efficiently, i.e., $\operatorname{poly}\big(n, \log(1/\varepsilon)\big)$
- **Null Cone Membership (NCM):**
  Given $v \in V$. *Decide* whether $\operatorname{cap}_G(v) = 0$.

Moment map and Kempf-Ness

- Fix $v \in V \backslash \{0\}$, consider $F_v \colon G \to \mathbb{R},\ g \mapsto \|g \cdot v\|^2$.
- **Moment map $\boldsymbol{\mu_G}$:**
  Think of $\mu_G(v)$ as the *gradient of* $\log F_v$ *at* $\mathrm{id} \in G$.

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  - can be used to decide NCM
- **Scaling Problem:** Given $v \in V$ and $\varepsilon > 0$.
  Compute $g \in G$ such that $\|\mu_G(g \cdot v)\| \leq \varepsilon$.
- Quantitative version of Kempf-Ness relates Norm Minimzation and the Scaling Problem, [BFG+19]

Motivation: Versatile Applications

- **Convex Optimization:** Unconstrained geometric programming
- **Algebraic Geometry:** constructing moduli spaces
- **Analysis:** Brascamp-Lieb inequalities
- **Physics:** quantum information theory
- **Computer Science:** non-commutative RIT
- **Geometric Complexity Theory**
- **(Algebraic) Statistics:** matrix scaling; ML estimation for Gaussian group models and for log-linear models [AKRS21]

Motivation: Versatile Applications

- $\mathrm{T} = \mathrm{ST}(n)^d$ commutative; $G = \mathrm{SL}(n)^d$ non-commutative
- natural action on $(\mathbb{C}^n)^{\otimes d}$ via $g_1 \otimes \cdots \otimes g_d$
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	T: commutative	G: non-commutative
$d = 2$	**matrix scaling:** statistics, bipartite matching, optimal transport, ...	**operator scaling:** non-commutative RIT, Tyler's M estimator, matrix normal models, ...
$d = 3$	**array scaling:** multimarginal transport, ...	**tensor scaling:** geometric complexity theory, quantum information theory, tensor normal models, ...

Digression: Matrix Scaling

- A non-negative matrix $A \in \mathbb{R}_{\geq 0}^{n \times n}$ is called **doubly stochastic**, if each of its row sums and each of its column sums equals one.
- For *positive diagonal* matrices X and Y, we say XAY is a **scaling** of A.
- **Matrix Scaling:** Given non-negative matrix A.
  Can we scale A arbitrarily close to doubly stochastic?

$$XAY = \begin{pmatrix} e^{x_1} & & \\ & \ddots & \\ & & e^{x_n} \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} e^{y_1} & & \\ & \ddots & \\ & & e^{y_n} \end{pmatrix}$$

where $x, y \in \mathbb{R}^n$.

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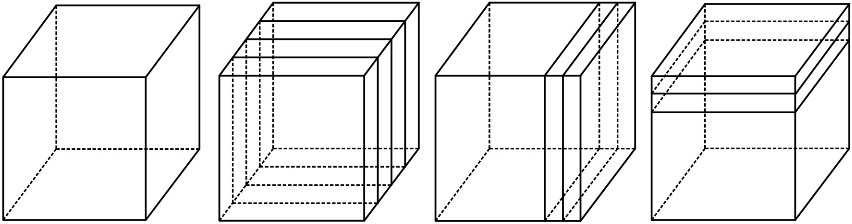
$$\begin{pmatrix} \varepsilon & & \\ & \varepsilon^{-1} & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon & & \\ & \varepsilon^{-1} & \\ & & 1 \end{pmatrix} = \begin{pmatrix} \varepsilon^2 & 1 & \varepsilon \\ 1 & 0 & 0 \\ \varepsilon & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Glimpse of Array Scaling

- Given an array $A \in (\mathbb{R}^n_{\geq 0})^{\otimes 3}$. Scaling of A is $(X \otimes Y \otimes Z)(A)$, where X, Y and Z are positive diagonal matrices.
- **Array Scaling:** Can we scale A in the limit to tristochastic?
- **tristochastic:** all slice sums are equal to one



Matrix Scaling as a geometric program

- For $\mathrm{T} = \mathrm{ST}(n)^2$ and a matrix $v \in \mathbb{C}^{n \times n}$ we get

$$\mathrm{cap}_{\mathrm{T}}(v) = \cdots = \inf_{x,y \in \mathbb{R}^n} f_v(x,y) := \inf_{x,y \in \mathbb{R}^n} \sum_{i,j=1}^{n} |v_{ij}|^2 \, e^{(\varepsilon_i, \varepsilon_j) \cdot (x,y)}$$

where $\varepsilon_i := e_i - \frac{1}{n} \mathbb{1}_n$. This is matrix scaling for $A = (|v_{ij}|^2)_{i,j}$!

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- $\|\mu_{\mathrm{T}}(g \cdot v)\| = \|\nabla \log \big(f_v(x,y)\big)\|$ measures the deviation of XAY from being doubly stochastic.

- **Scaling Problem:** (closely related to Norm Minimization)
  Compute scaling of A that is ε-close to doubly stochastic.

Matrix Scaling: important concepts

- The **weight polytope** of v is $\Delta_{\mathrm{T}}(v) := \operatorname{conv}\{(\varepsilon_i, \varepsilon_j) \mid v_{ij} \neq 0\}$.
- **Fact:** $\operatorname{cap}_{\mathrm{T}}(v) > 0 \quad \Leftrightarrow \quad 0 \in \Delta_{\mathrm{T}}(v)$.
- The function $(x, y) \mapsto \log\big(f_v(x, y)\big)$ is convex.
- $\|\mu_{\mathrm{T}}(g \cdot v)\| = \|\nabla \log\big(f_v(x, y)\big)\|$ measures the deviation of XAY from being doubly stochastic.

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- $\|\mu_{\mathrm{T}}(g \cdot v)\| = \|\nabla \log\big(f_v(x, y)\big)\|$ measures the deviation of XAY from being doubly stochastic.
- Concepts generalize to $d = 3$ and to G non-commutative
- For $d = 3$, array scaling arises for a tensor $v = (v_{ijk})_{i,j,k}$ as

$$\operatorname{cap}_{\mathrm{T}}(v) = \inf_{x,y,z \in \mathbb{R}^n} f_v(x, y, z) := \inf_{x,y,z \in \mathbb{R}^n} \sum_{i,j,k=1}^{n} |v_{ijk}|^2 \, e^{(\varepsilon_i, \varepsilon_j, \varepsilon_k) \cdot (x,y,z)}$$

Questions?

Next: 2. Scaling Algorithms and their Complexity

Scaling Algorithms and their Complexity

HP: efficient computation of high precision solutions, i.e., in $\mathrm{poly}\left(n, \log(1/\varepsilon)\right)$ time, for norm minimization problem

NCM: null cone membership solvable in poly-time

	T: commutative	G: non-commutative
$d = 2$	**matrix scaling:** HP, NCM (trust region, IPM)	**operator scaling:** HP, NCM (trust region)
$d = 3$	**array scaling:** HP, NCM (IPM, *not* via trust region)	**tensor scaling:** HP, NCM (no IPM available)

IPM is a shorthand for interior point methods

Natural Questions

Can we explain the dichotomy between $d = 2$ and $d = 3$?

- Why do gradient descent and trust region methods not suffice for HP and NCM when $d = 3$?
- Are known algorithms good enough for tensor scaling and only the complexity analysis lacks to show this?
  Or do we need new algorithmic approaches?

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  Or do we need new algorithmic approaches?

To answer these questions we investigate

- for HP: *diameter* of approximate minimizers
- for NCM: *precision parameters* (*weight margin*, *gap*)

Questions?

Next: 3. Bounds on Complexity Parameters

Part I: Diameter

Diameter

Norm minimization: Given a tensor $v = (v_{ijk})_{i,j,k}$ and $\varepsilon > 0$.
Find $g \in G$ such that $\|g \cdot v\|^2 \leq \mathrm{cap}(v) + \varepsilon$.

Think of the **diameter** $D_v(\varepsilon)$ as

- (bit) complexity of approximate minimizer
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In the commutative case use

$$\mathrm{cap}(v) = \inf_{x,y,z \in \mathbb{R}^n} f_v(x,y,z) := \inf_{x,y,z \in \mathbb{R}^n} \sum_{i,j,k=1}^{n} |v_{ijk}|^2 \, e^{(\varepsilon_i, \varepsilon_j, \varepsilon_k) \cdot (x,y,z)}$$

and define $\boldsymbol{D_v(\varepsilon)}$ as the infimum over all $R > 0$ such that

$$\inf_{\|(x,y,z)\| \leq R} f_v(x,y,z) \leq \mathrm{cap}(v) + \varepsilon.$$

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Known upper bounds on $D_v(\varepsilon)$:

	T: commutative	G: non-commutative
$d = 2$	**matrix scaling:** $O(n \log(1/\varepsilon))$	**operator scaling:** $O\big(n^{3/2} \log(1/\varepsilon)\big)$
$d = 3$	**array scaling:** [SV19] $O\big(n^{3/2} 2^{6n} \log(1/\varepsilon)\big)$	**tensor scaling:** [BFG+19] $O\big(2^{n \log(n)} \log(1/\varepsilon)\big)$

Diameter: Main Theorem

Theorem: Diameter bound for tensor scaling [Franks, R.]

There exists a constant $C > 0$ such that for all $\varepsilon \leq \exp(-Cn^2 \log n)$, there is a tensor $v = v(\varepsilon)$ with

$$D_v(\varepsilon) = \Omega\left(2^{n/3} \log(1/\varepsilon)\right).$$

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Here, v is **reasonably nice**:
v has $O(n)$ nonzero entries of bit complexity $O(\log n + \log(1/\varepsilon))$, $1/4 \leq \operatorname{cap}(v) \leq 1$ and $1/2 \leq \|v\| \leq 1$.

A similar theorem holds for array scaling (commutative and $d = 3$).

$\rightsquigarrow$ There we can drop the dependence of v on ε.

Diameter: Implications of Main Theorem

- algorithms with constant (or poly-bounded!) step size cannot provide HP for array/tensor scaling, e.g.,
  - gradient descent
  - trust region methods
- **commutative** (array scaling): our construction answers an open problem of [SV19] in the affirmative
- **non-commutative** (tensor scaling): together with a result of [HM21] our construction suggests that even cutting plane methods (as described by [Rus20]) do not suffice for HP

Questions?

Next: 3. Bounds on Complexity Parameters

Part II: Precision Parameters

Precision Parameters

- **NCM:** For $v \in V$, decide whether $\mathrm{cap}(v) = 0$.
- "global" parameters $\gamma_{\mathrm{T}}(\pi)$ (**weight margin**) and $\gamma_G(\pi)$ (**gap**)
  - capture required precision to solve NCM for T resp. G
  - always have $\gamma_{\mathrm{T}}(\pi) \leq \gamma_G(\pi)$

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  - always have $\gamma_{\mathrm{T}}(\pi) \leq \gamma_G(\pi)$
- $\gamma_{\mathrm{T}}(\pi)^{-1}$ appears polynomially in running time bounds, [BFG+19]
- Diameter bound: $D_v(\varepsilon) \leq \operatorname{poly}\left(\gamma_{\mathrm{T}}(\pi)^{-1}, \log(1/\varepsilon)\right)$, [BFG+19]
- **Quantitative Kempf-Ness [BFG+19]:** For $v \neq 0$

$$1 - \frac{\|\mu_G(v)\|_F}{\gamma_{\mathrm{T}}(\pi)} \leq \frac{\operatorname{cap}_G(v)}{\|v\|^2} \left(\leq 1 - \frac{\|\mu_G(v)\|^2}{4\,\mathrm{N}(\pi)^2} \right).$$

Weight margin: Formal Definition

- **Precision Parameter:**
  - $\gamma_{\mathrm{T}}(\pi) := \min \left\{ \|\mu_{\mathrm{T}}(v)\| \mid v \neq 0,\ \mathrm{cap}_{\mathrm{T}}(v) = 0 \right\}$
  - $\gamma_{\mathrm{T}}(\pi)$ is the largest constant with the property: Whenever $\|\mu_{\mathrm{T}}(t \cdot v)\| < \gamma_{\mathrm{T}}(\pi)$, then $\mathrm{cap}_{\mathrm{T}}(v) > 0$.

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- **Combinatorial Description:**
  Matrix scaling: set of weights is $\Omega = \{(\varepsilon_i, \varepsilon_j) \mid i, j \in [n]\} \subseteq \mathbb{R}^{2n}$

$$\begin{aligned} \gamma_{\mathrm{T}}(\pi) &= \min \left\{ \operatorname{dist}\big(0, \operatorname{conv}(S)\big) \mid S \subseteq \Omega, 0 \notin \operatorname{conv}(S) \right\} \\ &= \min \left\{ \operatorname{dist}\big(0, \Delta_{\mathrm{T}}(v)\big) \mid v \neq 0,\ 0 \notin \Delta_{\mathrm{T}}(v) \right\} \end{aligned}$$

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- Analogously for $\gamma_G(\pi)$ via μ_G and moment polytopes $\Delta_G(v)$

Weight margin and Gap: Interesting Example

Let $\tau_{n,d}$ be the natural action of $G = \mathrm{SL}(n)^d$ on the quiver Q_d with dimension vector $(n, \ldots, n)$, where Q_d is

$$1 \longleftarrow 2 \longrightarrow 3 \cdots\cdots\cdots\; d-2 \longrightarrow d-1 \longleftarrow d.$$

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Theorem: small Weight margin and large Gap

For $n, d \geq 2$ it holds that

(a) $\gamma_{\mathrm{T}}(\tau_{n,d}) \leq (n-1)^{-d+1}$ [Franks, R.]

(b) $\gamma_G(\tau_{n,d}) \gtrsim (dn)^{-1}$ [Franks, Makam]

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(c) $\gamma_G\left(\tau_{n,d}^{\oplus n}\right) \leq (n-1)^{-d+1}$ [Franks, R.]

Note: $\tau_{n,d}^{\oplus k}$ corresponds to the action of G on Q_d, but each arrow appears k-many times.

Weight margin and Gap: Main Theorem

Known lower bounds:

- for $d = 2$: $\Omega\big(n^{-3/2}\big) = \gamma_{\mathrm{T}}(\pi_{n,2}) \le \gamma_G(\pi_{n,2})$ [LSW98; Gur04]
- for $d = 3$: $\Omega\big((\sqrt{3}n)^{-3n-1}\big) = \gamma_{\mathrm{T}}(\pi_{n,3}) \le \gamma_G(\pi_{n,3})$ [BFG+19]

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Theorem: Bound for array/tensor scaling [Franks, R.]

There is a constant $C > 0$ such that for all $n \geq 2$, $d \geq 3$

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Bound explains:

- why current methods cannot give NCM for tensor scaling
- why diameter can be exponentially large for array/tensor scaling

Weight margin and Gap: Detailed Bounds

Theorem: Detailed bounds [Franks, R.]

(a) For $n = 2$ and $d \geq 3$, $\gamma_{\mathrm{T}}(\pi_{2,d}) \leq \gamma_G(\pi_{2,d}) \leq 2^{-\frac{d}{2}+1} = 2^{-\frac{nd}{4}+1}$.

- [AV97]: $\gamma_{\mathrm{T}}(\pi_{2,d}) = 2^{-\Theta(d \log d)}$
  - not easy to check whether their construction is *free*
- [MS18]: Qubits $v \in (\mathbb{C}^2)^{\otimes d}$ witnessing the exponentially small behaviour of $\gamma_G(\pi_{2,d})$ are rare

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(b) For $n \geq 3$ and $d = 3$, $\gamma_{\mathrm{T}}(\pi_{n,3}) \leq \gamma_G(\pi_{n,3}) \leq 2^{-n+1} = 2^{-\frac{nd}{3}+1}$.

(c) If $n \geq 3$ and $d = 6r - 3$ for some integer $r \geq 2$, then

$$\gamma_{\mathrm{T}}(\pi_{n,d}) \leq \gamma_G(\pi_{n,d}) \leq 2^{-r(n-1)+1} \approx 2^{-\frac{nd}{6}}.$$

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(a) For $n = 2$ and $d \geq 3$, $\gamma_{\mathrm{T}}(\pi_{2,d}) \leq \gamma_G(\pi_{2,d}) \leq 2^{-\frac{d}{2}+1} = 2^{-\frac{nd}{4}+1}$.

(b) For $n \geq 3$ and $d = 3$, $\gamma_{\mathrm{T}}(\pi_{n,3}) \leq \gamma_G(\pi_{n,3}) \leq 2^{-n+1} = 2^{-\frac{nd}{3}+1}$.

(c) If $n \geq 3$ and $d = 6r - 3$ for some integer $r \geq 2$, then

$$\gamma_{\mathrm{T}}(\pi_{n,d}) \leq \gamma_G(\pi_{n,d}) \leq 2^{-r(n-1)+1} \approx 2^{-\frac{nd}{6}}.$$

- Above results can be padded to obtain previous theorem.
  - constant $C > 0$ is reasonable (think of $C \approx \frac{1}{6}$ for $n, d \gg 0$)
- can be used to obtain exponentially small upper bounds on $\gamma_{\mathrm{T}}(\pi)$ and $\gamma_G(\pi)$ for *polynomial scaling* [Franks, R.]

Questions?

Next: 4. Glimpse of main Proof Techniques

Proof Ideas: commutative case

- Both, diameter bounds and weight margin bounds are proven by exploiting **extremal combinatorics** (of set of weights $\Omega(\pi_{n,d})$)
- **Intuition for $\boldsymbol{\gamma_{\mathrm{T}}(\pi)}$:** Birkhoff polytope B_2 of doubly stochastic matrices behaves well, while its 3-dim'l analogue B_3 does not.

$$B_2 = \left\{ (x_{ij}) \in \left(\mathbb{R}^{n \times n}_{\geq 0} \right) \mid \forall i_0, j_0 \in [n] \colon \sum_j x_{i_0,j} = 1, \sum_i x_{i,j_0} = 1 \right\}$$

$$B_3 = \left\{ (x_{ijk}) \in \left(\mathbb{R}^{n}_{\geq 0} \right)^{\otimes 3} \mid \forall i_0 \in [n] \colon \sum_{j,k} x_{i_0,j,k} = 1 \text{ etc.} \right\}$$

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- **E.g., for $\boldsymbol{\gamma_{\mathbf{T}}(\pi_{n,3})}$:** we crucially use a result of [Kra07], that explicitly states a vertex of B_3 with an exponentially small entry $\rightsquigarrow$ gives $\Gamma_{n,3} \subseteq \Omega(\pi_{n,3})$ with $0 < \operatorname{dist}\big(0, \operatorname{conv}(\Gamma_{n,3})\big) \leq 2^{-n+1}$

Proof Ideas: non-commutative case

- **Problem:** We act by general (and not just diagonal) matrices.
- **Solution:** Notion of a *free* tensor

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Definition: *free* tensor $v = (v_{ijk})_{i,j,k}$

Whenever $v_{ijk} \neq 0$ and $v_{i'j'k'} \neq 0$, then the triples (i, j, k) and (i', j', k') are *either* the same *or* differ in at least two entries.

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- appears as *strong orthogonal* in [DK85]
- used to study moment maps and/or moment polytopes, e.g., in [Sja98], [Fra02], [MS15] and [CVZ18]

Freeness for 3-tensors

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- Given $v \in (\mathbb{C}^n)^{\otimes 3} \setminus \{0\}$, let $M^{(1)}, M^{(2)}, M^{(3)} \in \mathbb{C}^{n \times n^2}$ be the *flattenings* of v, e.g., $M^{(1)}_{i,(j,k)} = v_{ijk}$. Then

$$\mu_G(v) = \left(M^{(l)} \left(M^{(l)} \right)^\dagger - \frac{\|M^{(l)}\|_F^2}{n} I_n \right)_{l=1,2,3}$$

- If v is free, then $\mu_G(v)$ diagonal. E.g., for $s,t \in [n]$, $s \neq t$

$$\left(M^{(1)} \left(M^{(1)} \right)^\dagger \right)_{s,t} = \sum_{j,k=1}^{n} M^{(1)}_{s,(j,k)} \overline{M^{(1)}_{t,(j,k)}} = \sum_{j,k=1}^{n} v_{s,j,k} \overline{v_{t,j,k}} \overset{\text{free}}{=} 0.$$

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Whenever $v_{ijk} \neq 0$ and $v_{i'j'k'} \neq 0$, then the triples (i,j,k) and (i',j',k') are *either* the same *or* differ in at least two entries.

- Freeness ensures that *"all information is contained in the action with diagonal matrices"*
  - can lift free constructions from the commutative to the non-commutative case
- All our constructions are free.

Questions?

Next: 5. Outlook and Open Problems

Outlook and Open Problems

- Is $\varepsilon \leq \exp(-Cn^2 \log n)$ necessary for our diameter bound? Or are there similar diameter bounds for, e.g., $\varepsilon \leq \exp(-Cn)$?
- Are the upper bounds on $\gamma_{\mathrm{T}}(\pi_{n,d})$ and $\gamma_G(\pi_{n,d})$ essentially tight?
- Large gap $\gamma_G(\pi)$ for *all* quivers of finite type?
- Quantitative Kempf-Ness via the gap $\gamma_G(\pi)$? (instead of $\gamma_{\mathrm{T}}(\pi)$)

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- Quantitative Kempf-Ness via the gap $\gamma_G(\pi)$? (instead of $\gamma_{\mathrm{T}}(\pi)$)
- **Upshot:** Our results show that for *tensor scaling* current methods *neither* suffice for HP *nor* for NCM.

  Therefore, our work strongly motivates:
- **Major Goal:** Find (interior point) methods for geodesic convex setting with running time $\mathrm{poly}\big(\boldsymbol{\log(1/\gamma(\pi))}, \log(1/\varepsilon)\big)$.

Thank you!

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Please find the full list of references in our paper!

Polynomial Scaling

Consider the scaling action on homogeneous polynomials of degree d:

$$\varrho_{n,d} \colon \ \mathrm{SL}(n) \to \mathrm{GL}\left(\mathbb{C}[x_1, \ldots, x_n]_d\right), \ g \mapsto \left(p(x) \mapsto p(g^{-1} \cdot x)\right)$$

As a consequence of the theorem for d-tensor scaling we obtain

Theorem: Gap for polynomial scaling [Franks, R.]

There exists a constant $C > 0$ such that for $d \geq 3$ and $n = dm$, where $m \geq 2$, one has

$$\gamma_{\mathrm{T}}(\varrho_{n,d}) \leq \gamma_G(\varrho_{n,d}) \leq 2^{-Cdm} = 2^{-Cn}.$$

More concretely, for $d = 3$ (i.e., *cubic forms*) and $m \geq 3$ it holds that

$$\gamma_{\mathrm{T}}(\varrho_{n,3}) \leq \gamma_G(\varrho_{n,3}) \leq 2^{-m+1} = 2^{-\frac{n}{3}+1}.$$

Bound on $\gamma_{\mathrm{T}}(\pi_{n,3})$: the witness set $\Gamma_{n,3}$

Recall: $\Omega(\pi_{n,3}) = \big\{(\varepsilon_i, \varepsilon_j, \varepsilon_k) \mid i, j, k \in [n]\big\}$

$\gamma_{\mathrm{T}}(\pi_{n,3}) = \min \big\{ \operatorname{dist}\big(0, \operatorname{conv}(S)\big) \mid S \subseteq \Omega(\pi_{n,3}),\ 0 \notin \operatorname{conv}(S) \big\}$

Want: $\Gamma_{n,3} \subseteq \Omega(\pi_{n,3})$ with $0 < \operatorname{dist}\big(0, \operatorname{conv}(\Gamma_{n,3})\big) \leq 2^{-n+1}$

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- [Kra07]: the *"3-dimensional Birkhoff polytope"*

  $B_{n,3} = \{(x_{ijk}) \in (\mathbb{R}^n_{\geq 0})^{\otimes 3} \mid \forall i_0 \in [n]\colon\ \sum_{j,k} x_{i_0,j,k} = 1 \text{ etc.}\}$

  has a vertex (λ_{ijk}) with $\lambda_{111} = 2^{-n+1}$, support $\{(1,1,1)\} \cup \mathfrak{W}_n$, where $\mathfrak{W}_n = \{(s,1,s), (s,s,1), (s-1,s,s) \mid s = 2, \ldots, n\}$.

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- choose $\Gamma_{n,3} := \{(\varepsilon_i, \varepsilon_j, \varepsilon_k) \mid (i,j,k) \in \mathfrak{W}_n\}$

- $\sum_i \varepsilon_i = 0 \quad \Rightarrow \quad 0 =$
  $2^{n-1}(\varepsilon_1, \varepsilon_1, \varepsilon_1) + \sum_{(i,j,k) \in \mathfrak{W}_n} \lambda_{ijk}(\varepsilon_i, \varepsilon_j, \varepsilon_k)$

  $\Rightarrow \mathrm{dist}\,(0, \mathrm{conv}(\Gamma_{n,3})) \leq 2^{-n+1}$

Proof of $0 \notin \mathrm{conv}(\Gamma_{n,3})$

$(*)$ $\sum_i \frac{1}{n} \varepsilon_i$ is the only convex combination of $\varepsilon_1, \ldots, \varepsilon_n$ giving zero.

- Assume $0 \in \mathrm{conv}(\Gamma_{n,3})$. Then there are $a_s, b_s, c_s \geq 0$ with
$$\sum_{s=2}^{n} \Big(a_s(\varepsilon_s, \varepsilon_1, \varepsilon_s) + b_s(\varepsilon_s, \varepsilon_s, \varepsilon_1) + c_s(\varepsilon_{s-1}, \varepsilon_s, \varepsilon_s) \Big) = 0 \in (\mathbb{R}^n)^3$$

- For $s = 2, \ldots, n$, apply $(*)$ to coeff. of ε_{s-1} in first component resp. to coeff. of ε_s in second, third component:

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- inductively: $a_s = b_s = 0$ and $c_s = n^{-1}$ for $s = 2, \ldots, n$ $\lightning$