

On defective fourfolds

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A brief introduction

Curves are never defective.

Classification of defective surfaces: Severi (1901), Palatini (1906), Terracini (1921), Chiantini–C. (2001).

Classification of defective threefolds: Scorza (1908), Chiantini–C. (2001, 2005).

Classification of 1–defective fourfolds: Scorza (1909).

Scorza's arguments are often longwinded and obscure, and his classification eventually contains some gaps. In this talk I want to explain the full classification of secant 1-defective fourfolds.

The classification depends on two basic invariants, the **fibre defect** and the **tangential defect**, which I will now introduce.

The fibre defect

$X \subseteq \mathbb{P}^r$ is an n -dimensional, irreducible, reduced, non-degenerate, projective variety.

$S(X)$ is the **1-secant variety** of X , of dimension $s(X)$.

$\sigma(X) = \min\{r, 2n + 1\}$ is the **expected dimension** of $S(X)$.

Then $s(X) \leq \sigma(X)$ and X is **(1)-defective** if $s(X) < \sigma(X)$.

Standard examples of defective varieties are **cones**.

\mathbb{S}_X is the abstract secant variety of X of dimension $2n + 1$, i.e., the Zariski closure of the set

$$\{([\rho_0, \rho_1], x) \in \text{Sym}^2(X) \times \mathbb{P}^r : \rho_0 \neq \rho_1, x \in \langle \rho_0, \rho_1 \rangle\}.$$

One has the surjective morphism $p_X : \mathbb{S}_X \rightarrow S(X) \subseteq \mathbb{P}^r$, i.e., the projection to the second factor.

$$f(x) = \dim(\mathbb{S}_x) - s(X) = 2n + 1 - s(X)$$

is the dimension of the general fibre of p_X . This is called the **fibre defect** of X .

If X is defective, certainly $f(X) > 0$.

The tangential defect and the species

Let $x \in S(X)$ be a general point. Then $x \in \langle p_0, p_1 \rangle$, with $p_0, p_1 \in X$ general. Consider the Zariski closure of the set

$$\{p \in \text{Reg}(X) : T_{X,p} \subseteq T_{S(X),x} = \langle T_{X,p_0}, T_{X,p_1} \rangle\}.$$

Γ_{X,p_0,p_1} is the union of all irreducible components of this set containing p_0 and p_1 . It is called the **tangential contact locus** of X at p_0, p_1 .

Facts: Γ_{X,p_0,p_1} is either irreducible or it consists of two irreducible components of the same dimension each containing one of the points p_0, p_1 . Moreover Γ_{X,p_0,p_1} is smooth at p_0, p_1 .

$$\gamma(X) := \dim(\Gamma_{X,p_0,p_1})$$

is called the **tangential defect** of X . Of course one has

$$\gamma(X) \leq n - 1.$$

Moreover, one has

$$\gamma(X) \geq f(x).$$

$k(X) = n - \gamma(X)$ is called (after Scorza) the **species** of X .

If X is defective one has

$$1 \leq k(X) \leq n - 1 \quad \text{and} \quad n - 1 \geq \gamma(X) \geq f(X) \geq 1,$$

in particular, for $n = 4$ we have

$$1 \leq k(X) \leq 3 \quad \text{and} \quad 3 \geq \gamma(X) \geq f(X) \geq 1.$$

The classification

Next I will state the classification results, which will be made according to the values of $f(X)$ and $k(X)$.

The first two results will be general, i.e., no restriction on the dimension n of X .

Then I will focus on fourfolds.

From now on I will assume the variety X to be defective.

The case $f(X) = n - 1$

Of course this forces $\gamma(X) = n - 1$, hence $k(X) = 1$.

Theorem 1

If $f(X) = n - 1$ then X is either a cone over a curve or a cone over the Veronese surface $V_{2,2}$ in \mathbb{P}^5 .

These varieties are of course singular.

The case $f(X) = n - 2$

Theorem 2

If $n \geq 4$ and $f(X) = n - 2$ then X is of one of the following types:

- (i) X is a cone over a non-defective surface ($\gamma(X) = n - 2$);
- (ii) X sits in a $(n + 1)$ -dimensional cone with vertex a $(n - 1)$ -space over a curve ($\gamma(X) = n - 2$);
- (iii) X sits in a $(n + 1)$ -dimensional cone with vertex a $(n - 2)$ -space over the Veronese surface $V_{2,2}$ in \mathbb{P}^5 ($\gamma(X) = n - 1$);
- (iv) X is a cone over the Veronese threefold $V_{3,2}$ in \mathbb{P}^9 or over a projection of $V_{3,2}$ in \mathbb{P}^8 or \mathbb{P}^7 ($\gamma(X) = n - 1$);
- (v) X is either a cone over the Segre variety $\text{Seg}(2, 2) \subset \mathbb{P}^8$ with vertex of dimension $n - 5$, or a cone over a hyperplane section of $\text{Seg}(2, 2)$ ($\gamma(X) = n - 2$).

Cones are of course defective, and also the other varieties in the above list are defective.

Moreover the varieties in Theorem 2 are all singular, except for $n = 4$ in case (v) if

$$X = \text{Seg}(2, 2) \subset \mathbb{P}^8.$$

Fourfolds: the remaining cases

Next I will focus on fourfolds with

$$f(X) = 1, \quad r \geq 9$$

The classification breaks up in seven remaining cases:

- scrolls in 3-spaces over a curve;
- $k(X) = 3$, irreducible tangential contact locus of dimension 1;
- $k(X) = 3$, reducible tangential contact locus of dimension 1;
- $k(X) = 2$, irreducible tangential contact locus of dimension 2;
- $k(X) = 2$, reducible tangential contact locus of dimension 2;
- $k(X) = 1$, irreducible tangential contact locus of dimension 3;
- $k(X) = 1$, reducible tangential contact locus of dimension 3.

Example 1

In \mathbb{P}^r , with $r \geq 9$, take a plane Π .

Consider an irreducible scroll surface Σ sitting in a $(r - 3)$ -space skew with Π .

Let C be the curve parametrizing the lines of Σ . Let $C \subset \Pi^*$ be an irreducible curve. Suppose there is a rational dominant map $\phi : C \dashrightarrow C$.

Consider the Zariski closure X of the union of the 3-spaces joining a general line of Σ corresponding to a point c of C to the line of Π corresponding to the point $\phi(c)$.

Assume that $C \subset \Pi^*$ is not a line, so that the rulings of X do not pass through the same point, hence X is not a cone.

Since the rulings of X pairwise meet at a point, then also the tangent spaces to X pairwise meet, hence X is defective by Terracini's Lemma. Moreover X is singular.

Defective fourfold scrolls: examples (II)

Example 2

Consider an irreducible 1–dimensional family \mathcal{P} of planes in \mathbb{P}^r , with $r \geq 9$, such that two general planes of the family intersect at a point, and it does not happen that either all the planes of \mathcal{P} pass through the same point or all the planes of \mathcal{P} intersect the same plane along a line.

By a result of U. Morin the planes of \mathcal{P} span at most a 5–space.

Consider an irreducible curve $C \subset \mathbb{P}^r$ which has a dominant rational map $\phi : C \dashrightarrow \mathcal{P}$, and is in a sufficiently general position with respect to the planes of \mathcal{P} .

Then consider the 4–dimensional scroll X which is the Zariski closure of the union of all 3–spaces joining a general point $x \in C$, with the plane corresponding to $\phi(x)$.

Then X is not a cone, it is defective and singular.

Example 3

Let $S \subset \mathbb{P}^9$ be an irreducible, projective surface, and let \mathcal{C} be an irreducible 1–dimensional family of curves on S not passing through the same point, such that the general curve in \mathcal{C} is irreducible, spans a 3–dimensional space and two general curves of \mathcal{C} intersect at a point.

Let X be the 4–fold scroll which is swept out by the 3–dimensional spaces spanned by the curves of \mathcal{C} .

Then X is not a cone and it is defective and singular.

For a specific example of a surface S of the above type, consider the Veronese surface $V_{2,3} \subset \mathbb{P}^9$. It contains the images of the lines of \mathbb{P}^2 which are rational normal cubics, each spanning a 3–dimensional linear space. Let \mathcal{C} be an irreducible 1–dimensional family of these curves corresponding to a family of lines which do not pass through the same point.

There are actually infinitely many examples of this type.

Defective fourfold scrolls: classification

Theorem 3

Let $X \subset \mathbb{P}^r$, $r \geq 9$, be an irreducible, non-degenerate, linearly normal, defective 4-fold scroll with $f(X) = 1$. Then one of the following cases occurs:

- (i) X is a cone with vertex a point over a non-defective threefold scroll (in planes over a curve);
- (ii) X is as in one of the Examples 1,2,3 above.

In all these cases X is singular.

Defective fourfolds X with $f(X) = 1$, $k(X) = 3$ and irreducible tangential contact locus

Theorem 4

Let $X \subset \mathbb{P}^r$, with $r \geq 9$, be a projective, non-degenerate, linearly normal, defective fourfold. Assume that $f(X) = 1$, $k(X) = 3$ and with irreducible general tangential contact locus. Then X is one of the following:

- (i) an internal projection of the Veronese 4-fold $V_{4,2} \subset \mathbb{P}^{14}$ from finitely many points and at most finitely many lines through its general point;
- (ii) the projection in \mathbb{P}^{11} of $V_{4,2} \subset \mathbb{P}^{14}$ from the plane spanned by a conic on it;
- (iii) the projection in \mathbb{P}^9 of $V_{4,2} \subset \mathbb{P}^{14}$ from a 4-space spanned by a rational normal quartic curve on it;
- (iv) a hyperplane section in \mathbb{P}^{10} of $\text{Seg}(2, 3) \subset \mathbb{P}^{11}$.

All these varieties can be smooth.

One checks that the variety in (iv) is also the internal projection of the variety in (ii) from a point. Since $V_{4,2}$ is well known to be defective then these varieties are defective as well.

Defective fourfolds X with $f(X) = 1$, $k(X) = 3$ and reducible tangential contact locus

In this case one proves that X is a scroll in 3–spaces over a curve, so it falls in the scroll classification.

Defective fourfolds X with $f(X) = 1$, $k(X) = 2$ and irreducible tangential contact locus

Theorem 5

Let $X \subset \mathbb{P}^r$, with $r \geq 9$, be a projective, non-degenerate, linearly normal, defective fourfold. Assume that $f(X) = 1$, $k(X) = 2$ and with irreducible general tangential contact locus. Then X is one of the following:

- (i) $r = 9$ and X sits in a cone with vertex a point over $\text{Seg}(2, 2)$ in \mathbb{P}^8 ;
- (ii) $r = 9$ and X sits in a cone with vertex a line over a hyperplane section of $\text{Seg}(2, 2)$ in \mathbb{P}^7 ;
- (iii) $r = 9$ and X sits in a cone with vertex a line over a (defective) 3-fold in \mathbb{P}^7 sitting in a cone with vertex a line over the Veronese surface $V_{2,2}$, in particular X sits in a 6-dimensional cone with vertex a 3-space over the Veronese surface $V_{2,2}$;
- (iv) $9 \leq r \leq 11$ and X sits in a cone with vertex a line over the Veronese 3-fold $V_{3,2}$ in \mathbb{P}^9 or a projection of it in \mathbb{P}^8 or \mathbb{P}^7 ;
- (v) X is swept out by a 4-dimensional family \mathcal{S} of surfaces spanning a 4-space, such that two general surfaces in \mathcal{S} intersect at a point; in this case the general surface in \mathcal{S} is rational and X itself is rational;
- (vi) X is swept out by a 3-dimensional family \mathcal{R} of lines and it is singular along a linear space Π of dimension ε , with $2 \leq \varepsilon \leq 3$, which is cut out in one point by the general line in \mathcal{R} , and X projects from Π to a 3-dimensional variety $Y \subset \mathbb{P}^{r-\varepsilon-1}$, which contains a 4-dimensional family \mathcal{C} of (generically irreducible) conics such that there is a conic in \mathcal{C} passing through two general points of Y and the counterimage of the general conic of \mathcal{C} via the projection from Π is a non-developable scroll spanning a 4-space with a line directrix sitting in Π . In any event $r \leq 10 + \varepsilon$.

It is easy to check that these varieties are defective.

All these varieties are singular, except possibly in case (i) and (v), though it is not known if fourfolds as in case (v) exist.

Defective fourfolds X with $f(X) = 1$, $k(X) = 2$ and reducible tangential contact locus

Theorem 6

Let $X \subset \mathbb{P}^r$, with $r \geq 9$, be a projective, non-degenerate, linearly normal, defective fourfold. Assume that $f(X) = 1$, $k(X) = 2$ and with reducible general tangential contact locus. Then X is either a cone or it sits in a 5-dimensional cone with vertex a plane over a surface.

These varieties are clearly defective and singular.

Defective fourfolds X with $f(X) = 1$, $k(X) = 1$ and irreducible tangential contact locus

Theorem 7

Let $X \subset \mathbb{P}^r$, with $r \geq 9$, be a projective, non-degenerate, linearly normal, defective fourfold. Assume that $f(X) = 1$, $k(X) = 1$ and with irreducible general tangential contact locus. Then X is either a cone or it sits in a 6-dimensional cone with vertex a 3-space over a Veronese surface $V_{2,2}$.

These varieties are clearly defective and singular.

Defective fourfolds X with $f(X) = 1$, $k(X) = 1$ and reducible tangential contact locus

Theorem 8

Let $X \subset \mathbb{P}^r$, with $r \geq 9$, be a projective, non-degenerate, linearly normal, defective fourfold. Assume that X is neither a cone nor a scroll and $f(X) = 1$, $k(X) = 1$, with reducible general tangential contact locus. Then one of the following occurs:

- (i) X lies in a 6-dimensional cone with vertex a 4-space over a curve;
- (ii) X lies in a 5-dimensional cone with vertex a plane over a scroll surface.

These varieties are clearly defective and singular.

Sketch of proof of Theorem 4: the case $f(X) = 1$, $k(X) = 3$ and irreducible tangential contact locus

Recall:

Theorem 4

Let $X \subset \mathbb{P}^r$, with $r \geq 9$, be a projective, non-degenerate, linearly normal, defective fourfold. Assume that $f(X) = 1$, $k(X) = 3$ and with irreducible general tangential contact locus. Then X is one of the following:

- (i) an internal projection of the Veronese 4-fold $V_{4,2} \subset \mathbb{P}^{14}$ from finitely many points and at most finitely many lines through its general point;
- (ii) the projection in \mathbb{P}^{11} of $V_{4,2} \subset \mathbb{P}^{14}$ from the plane spanned by a conic on it;
- (iii) the projection in \mathbb{P}^9 of $V_{4,2} \subset \mathbb{P}^{14}$ from a 4-space spanned by a rational normal quartic curve on it;
- (iv) a hyperplane section in \mathbb{P}^{10} of $\text{Seg}(2, 3) \subset \mathbb{P}^{11}$.

Sketch of proof of Theorem 4 (I): the Scorza's map

In this case one has

$$f(X) = \gamma(X) = 1$$

and the general tangential contact locus is irreducible.

One checks that if this happens then the general tangential contact locus is a **conic**. So X possesses a 6-dimensional family \mathcal{C} of conics such that given two general points p_0, p_1 in X there is a unique conic in \mathcal{C} containing p_0, p_1 .

The first basic result is the following:

Lemma

In the above setting, let $x \in X$ be a general point and let t be a general tangent direction to X at x . Then there is a unique irreducible conic of the family \mathcal{C} passing through x and tangent to t .

If $x \in X$ is a general point, define the **Scorza's map**

$$S_x : X \dashrightarrow T_{X,x}$$

sending a general point $y \in X$ to the point $T_{X,x} \cap T_{X,y}$.

Proposition

In the above setting, Scorza's map S_x is birational, hence X is rational. Moreover its inverse S_x^{-1} is defined by a linear system of hypersurfaces of degree d having a point of multiplicity $d - 2$ at x .

Sketch of proof of Theorem 4 (II)

Let $x \in X$ be a general point. Let

$$p_x : \mathcal{F}_x \rightarrow \mathcal{C}_x$$

be the flat family of conics in \mathcal{C} passing through x .

One has $\mathcal{F}_x \subseteq \mathcal{C}_x \times X$. Let

$$\phi_x : \mathcal{F}_x \rightarrow X$$

be the projection to the second factor, which is birational. Then \mathcal{F}_x is irreducible, so also \mathcal{C}_x is irreducible with

$$\dim(\mathcal{C}_x) = 3 \quad \text{and} \quad \dim(\mathcal{F}_x) = 4.$$

The map p_x has a natural section σ_x mapping a point of \mathcal{C}_x corresponding to a conic C , to the point $x \in C \subset \mathcal{F}_x$.

The image \mathcal{E}_x of σ_x is contracted by ϕ_x to the point x .

Consider the blow-up $\pi_x : \tilde{X} \rightarrow X$ of X at x . So we have the commutative diagram

$$\begin{array}{ccc} \mathcal{F}_x & \xrightarrow{\psi_x} & \tilde{X} \\ \downarrow p_x & \searrow \phi_x & \downarrow \pi_x \\ \mathcal{C}_x & & X \end{array}$$

σ_x (curved arrow from \mathcal{C}_x to \mathcal{F}_x)

where ψ_x is a birational map whose indeterminacy locus is contained in \mathcal{E}_x .

Sketch of proof of Theorem 4 (III)

$$\begin{array}{ccc}
 \mathcal{F}_x & \xrightarrow{\psi_x} & \tilde{X} \\
 \downarrow \rho_x & \searrow \phi_x & \downarrow \pi_x \\
 \mathcal{C}_x & & X
 \end{array}$$

σ_x (curved arrow from \mathcal{C}_x to \mathcal{F}_x)

Let S_x be the closed subscheme of \mathcal{C}_x parametrizing the conics in \mathcal{C}_x singular at x . Denote by \mathcal{D}_x the image of S_x via the section σ_x . Then \mathcal{D}_x is exactly the indeterminacy locus of ψ_x .

Thus ψ_x is defined at the general point of \mathcal{E}_x and its inverse is defined at the general point of the exceptional divisor $E_x \cong \mathbb{P}^3$ of the blow-up $\pi_X : \tilde{X} \rightarrow X$.

Hence ψ_x induces a birational map

$$\psi_{x,0} : \mathcal{E}_x \dashrightarrow E_x$$

The geometric meaning of which is clear: it associates to a point $(c, x) \in \mathcal{E}_x \setminus \mathcal{D}_x$ the point of E_x corresponding to the direction of the tangent line at x to the conic C corresponding to the point $c \in \mathcal{C}_x$.

Let D_x be the closure in E_x of the points corresponding to all directions at x of irreducible components of singular conics in S_x . Set

$$\chi(X) = \dim(D_x).$$

Sketch of proof of Theorem 4 (IV)

Assume $\chi(X) \leq 0$.

A crucial fact is that if $M \subset E_x \cong \mathbb{P}^3$ is a general line, then $L = \psi_{x,0}^{-1}(M)$ is a smooth irreducible rational curve in E_x such that for every point $(c, x) \in L$ the conic C corresponding to c is smooth.

By base change we obtain a surface S with a morphism

$$\beta : S \rightarrow L \cong \mathbb{P}^1$$

such that every fiber F of β is irreducible and maps to a conic in X through x via the tautological morphism $\phi_S : S \rightarrow X$.

We set $Y = \phi_S(S)$. The surface S is isomorphic to a surface \mathbb{F}_e , with $e \geq 0$. The section E_0 with $E_0^2 = -e$ is contracted to x on Y , hence $e \geq 1$.

The following result is basic:

Lemma

In the above set up (i.e., $\chi(X) \leq 0$), then $e = 1$ and Y is a Veronese surface $V_{2,2}$ in \mathbb{P}^5 .

Corollary

Let $X \subset \mathbb{P}^r$, with $r \geq 9$, be an irreducible, non-degenerate, defective variety, of dimension 4. Assume that $f(X) = 1$, $k(X) = 3$, the general tangential contact locus is irreducible and $\chi(X) \leq 0$. Then $r \leq 14$ and X is a projection to \mathbb{P}^r of the Veronese variety $V_{4,2}$.

Proof. By the Lemma, the restriction of S_x^{-1} to a general plane Π through x maps Π to a Veronese surface $V_{2,2}$.

This implies that S_x^{-1} is defined by a linear system of quadrics (i.e., $d = 2$), whence the claim follows. \diamond

Interlude: tangential projection and second fundamental form

Let $x \in X \subset \mathbb{P}^r$, with $r \geq 9$, be a general point, and consider:

- the **tangential projection** $\tau_X : X \dashrightarrow X_1 \subset \mathbb{P}^{r-5}$ from $T_{X,x}$.

The conics in \mathcal{C} through x are the fibres of τ_X , hence X_1 has dimension 3.

If $\tilde{\tau}_X : \tilde{X} \dashrightarrow X_1$ is a partial resolution of indeterminacies of τ_X , with $\tilde{X} \rightarrow X$ the blow-up of X at x , with exceptional divisor $E_X \cong \mathbb{P}^3$, then $\tilde{\tau}_X$ induces a dominant rational map $\tau'_X : E_X \dashrightarrow X_1$;

- the **second fundamental form** Π_x , that is a linear system of quadrics in E_x of dimension $r - 5$ that defines $\tau'_X : E_X \dashrightarrow X_1$.

Let B_x be the base locus scheme of Π_x .

Let L_x be the locus of points of E_x corresponding to tangent directions of the lines on X passing through x .

One has

$$L_x \subseteq B_x$$

and set

$$b(X) = \dim(B_x) \quad \text{and} \quad \ell(X) = \dim(L_x), \quad \text{hence} \quad \ell(X) \leq b(X).$$

Sketch of proof of Theorem 4 (V)

Proposition

Let $X \subset \mathbb{P}^r$, with $r \geq 9$, be a non-degenerate, defective variety, of dimension 4. Assume that $f(X) = \gamma(X) = 1$ and the general tangential contact locus is an irreducible conic. Then $b(X) \leq 1$ and if the equality holds then, for a general point $x \in X$, one has:

- (i) either B_x is a smooth conic and $r = 9$;
- (ii) or B_x is a line and $r \leq 11$;
- (iii) or B_x is the union of a line L and of a point $p \notin L$, and $r \leq 10$.

Proof. Recall that Π_x is a linear system of quadrics in $E_x \cong \mathbb{P}^3$ and $\dim(\Pi_x) = r - 5 \geq 4$. So Π_x cannot have a base locus of dimension 2, and if $b(X) = 1$, then the 1-dimensional components of B_x can be either a conic or a line. The assertion follows. \diamond

Sketch of proof of Theorem 4 (VI)

Now I can finish the proof of Theorem 4.

If $\ell(X) \leq 0$, then also $\chi(X) \leq 0$. Then S_X^{-1} is defined by a linear system Λ of quadrics in \mathbb{P}^4 and the base locus B of Λ must be finite because $\ell(X) \leq 0$, thus X an internal projection of the Veronese 4-fold $V_{4,2} \subset \mathbb{P}^{14}$ from finitely many points (case (i) of Theorem 4).

Assume $\ell(X) > 0$. Since $b(X) \geq \ell(X)$, by the Proposition in the previous slide we have

$$b(X) = \ell(X) = 1.$$

Hence there is an irreducible component \mathcal{R} of the Hilbert scheme of lines in X of dimension 4. By a result by E. Rogora, we have the following possibilities:

- (1) either X is swept out by a 1-dimensional family \mathcal{Q} of 3-dimensional quadrics of rank 5;
- (2) or X is swept out by a 2-dimensional family \mathcal{P} of planes;

and the lines of \mathcal{R} are the lines contained in the quadrics of \mathcal{Q} in case (1), the lines contained in the planes of \mathcal{P} in case (2).

Moreover, by the Proposition in the previous slide, there is only one quadric Q of the family \mathcal{Q} passing through the general point $x \in X$ in case (1), and in case (2) we have only one plane P of the family \mathcal{P} passing through x .

Sketch of proof of Theorem 4 (VII)

Suppose S_x^{-1} is defined by a linear system Λ of quadrics in $T_{X,x} \cong \mathbb{P}^4$ with base locus B , which is the case if $\chi(X) \leq 0$.

If $z \in \mathbb{P}^4$ is a general point, mapped by S_x^{-1} to a general point $x \in X$, then Π_x is projectively equivalent to the projection in \mathbb{P}^3 from z of the linear system of cones in Λ with vertex in z .

It follows that:

- (a) either B is a smooth conic;
- (b) or B is a line;
- (c) or B is the union of a line L and of a point $p \notin L$.

Case (a) leads to the case in which X is the projection of $V_{4,2} \subset \mathbb{P}^{14}$ from a 4-space spanned by a rational normal quartic curve on it (case (iii) of Theorem 4).

Case (b) leads to the case in which X is the projection of $V_{4,2} \subset \mathbb{P}^{14}$ from the plane spanned by a conic on it (case (ii) of Theorem 4).

Case (c) leads to the case in which X is the internal projection of X in case (ii) from a point, which is the same as a hyperplane section of $\text{Seg}(2, 3) \subset \mathbb{P}^{11}$ (case (iv) of Theorem 4).

Finally a detailed analysis shows that in any event S_x^{-1} is defined by a linear system Λ of quadrics in $T_{X,x} \cong \mathbb{P}^4$ and this ends the proof.

Thank you for your attention!